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Standard orthogonal vectors in semilinear spaces and their applications ☆

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ABSTRACT

This paper investigates the standard orthogonal vectors in semilinear spaces of n-dimensional vectors over commutative zerosumfree semirings. First, we discuss some characterizations of standard orthogonal vectors. Then as applications, we obtain some necessary and sufficient conditions that a set of vectors is a basis of a semilinear subspace which is generated by standard orthogonal vectors, prove that a set of linearly independent nonstandard orthogonal vectors cannot be orthogonalized if it has at least two nonzero vectors, and show that the analog of the Kronecker–Capelli theorem is valid for systems of equations.

1. Introduction

The study of semilinear structures over zerosumfree semirings has a long history. In 1979, Cuninghame-Green built a theory similar to that of linear algebra in min-plus algebra, for instance in [5], systems of linear equations, eigenvalue problems, independence, rank and dimension. Since then, a number of works on semilinear structure over zerosumfree semirings were published (see e.g. [2–4,10,11]). In 2007, Di Nola et al. used the notions of semirings and semimodule to introduce the concept of semilinear space in the MV-algebraic setting, and obtained some similar results as those of classical linear algebras (see [7]). In 2010, Perfilieva and Kupka showed that the necessary condition of the Kronecker–Capelli theorem is valid for systems of equations in a semilinear space of n-dimensional vectors (see [12]), Zhao and Wang gave a sufficient condition that each basis in semilinear spaces of

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n-dimensional vectors has the same number of elements over commutative zerosumfree semirings (see [17]), moreover, in 2011, they obtained a necessary and sufficient condition that each basis has the same number of elements over join-semirings (see [18]), where a join-semiring is just a kind of zerosumfree semiring. In 2011, Shu and Wang showed some necessary and sufficient conditions that each basis has the same number of elements over commutative zerosumfree semirings and proved that a set of vectors is a basis if and only if they are standard orthogonal (see [15]). In this paper, we further investigate the standard orthogonal vectors in semilinear spaces of n-dimensional vectors over commutative zerosumfree semirings, and discuss their characterizations, as applications, we first study the conditions that a set of vectors is a basis of a semilinear subspace which is generated by standard orthogonal vectors, and then prove that the analogue of the Kronecker–Capelli theorem is valid for systems of equations.

The paper is organized as follows. For the sake of convenience, some notions and previous results are given in Section 2. Section 3 discusses some properties of standard orthogonal vectors, obtains some sufficient and necessary conditions that a set of vectors is a basis of a semilinear subspace which is generated by standard orthogonal vectors, and proves that a set of linearly independent nonstandard orthogonal vectors cannot be orthogonalized if it has at least two nonzero vectors. Section 4 shows that the analogue of the Kronecker–Capelli theorem is valid for systems of equations. A conclusion is included in Section 5.

2. Previous results

In this section, we give some definitions and preliminary lemmas.

Definition 2.1 (Golan [8], Zimmermannn [19]). A semiring $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ is an algebraic structure with the following properties:

(i) $(L, +, 0)$ is a commutative monoid,

(ii) $(L, \cdot, 1)$ is a monoid,

(iii) $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(s + t) \cdot r = s \cdot r + t \cdot r$ hold for all $r, s, t \in L$,

(iv) $0 \cdot r = r \cdot 0 = 0$ holds for all $r \in L$,

(v) $0 \neq 1$.

A semiring $\mathcal{L}$ is commutative if $r \cdot r' = r' \cdot r$ for all $r, r' \in L$. A semiring $\mathcal{L}$ is called zerosumfree if $a + b = 0$ implies that $a = b = 0$ for any $a, b \in L$.

Example 2.1. Let $\mathbb{R}$ be the set of all real numbers. Then $\mathcal{L} = \langle \mathbb{R} \cup \{-\infty\}, +, \cdot, -\infty, 0\rangle$ is a semiring, where $a + b = \max\{a, b\}$ and $a \cdot b = a + b$ for $a, b \in \mathbb{R} \cup \{-\infty\}$ in which the last $+$ stands for the usual addition of real numbers.

Note that the semiring $\mathcal{L} = \langle \mathbb{R} \cup \{-\infty\}, +, \cdot, -\infty, 0\rangle$ is usually called a max-plus algebra or a schedule algebra (see e.g. [1,2,6]).

Example 2.2. Let $\mathbb{N} = \{0, 1, \ldots, n, \ldots\}$ be the set of all natural numbers. Then $\mathbb{N}$ with the usual operations of addition and multiplication of integers is a commutative zerosumfree semiring.

Example 2.3 (Zhao and Wang [17]). The following are examples of commutative zerosumfree semirings:

(i) The real interval $[0, 1]$ under the operations $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in [0, 1]$;

(ii) The nonnegative real numbers with the usual operations of addition and multiplication;

(iii) The nonnegative integers under the operations $a + b = \text{g.c.d.}\{a, b\}$ and $a \cdot b = \text{l.c.m.}\{a, b\}$ for nonnegative integers a and b, and g.c.d. (resp. l.c.m.) stands for the greatest (resp. least) common divisor (resp. multiple) between a and b.

Definition 2.2 (Zimmermannn [19]). Let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be a semiring and let $\mathcal{A} = \langle A, +_A, 0_A\rangle$ be a commutative monoid. If $^* : L \times A \to A$ is an external multiplication such that

(i) $(r \cdot r') * a = r*(r' * a)$,

(ii) $r * (a +_A a') = r * a +_A r * a'$,

(iii) $(r + r') * a = r * a +_A r' * a$,

(iv) $1 * a = a$,

(v) $0 * a = r * 0_A = 0_A$

for all $r, r' \in L$ and $a, a' \in A$ then $\langle \mathcal{L}, +, \cdot, 0, 1; *; A, +_A, 0_A\rangle$ is called a left $\mathcal{L}$-semimodule. The definition of right $\mathcal{L}$-semimodule is analogous, where the external multiplication is defined as a function $A \times L \to A$.

The following definition is a general version of that of a semilinear space in [7]:

Definition 2.3. Let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be a semiring. Then a semimodule over $\mathcal{L}$ is called an $\mathcal{L}$-semilinear space.

Note that in Definition 2.3, a semimodule stands for a left $\mathcal{L}$-semimodule or a right $\mathcal{L}$-semimodule as in [7]. Elements of an $\mathcal{L}$-semilinear space will be called vectors and elements of a semiring scalars. The former will be denoted by bold letters to distinguish them from scalars.

Without loss of generality, in what follows, we consider left $\mathcal{L}$-semimodules for convenience of notation. Let $\underline{n} = \{1, \ldots, n\}$. Then we can construct an $\mathcal{L}$-semilinear space as follows.

Example 2.4 (Shu and Wang [15]). (a) Let $\mathcal{L}$=$\langle L, +, \cdot, 0, 1\rangle$ be a semiring. For each $n \geqslant 1$, let

$$V_n(L) = \{(a_1, a_2, \ldots, a_n)^T : a_i \in L, i \in \underline{n}\}.$$

Define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^T,$$

$$r*\mathbf{x} = (r \cdot x_1, r \cdot x_2, \ldots, r \cdot x_n)^T$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T, \mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in V_n(L)$ and $r \in L$, where $(x_1, x_2, \ldots, x_n)^T$ denotes the transpose of $(x_1, x_2, \ldots, x_n)$. Then $\mathcal{V}_n = \langle L, +, \cdot, 0, 1; *; V_n(L), +, \mathbf{0}_{n\times 1}\rangle$ is an $\mathcal{L}$-semilinear space with $\mathbf{0}_{n\times 1} = (0, 0, \ldots, 0)^T$. Similarly, we can define the operations of addition and external multiplication on row vectors and obtain that $\mathcal{V}^n = \langle L, +, \cdot, 0, 1; *; V^n(L), +, \mathbf{0}_{1\times n}\rangle$ is also an $\mathcal{L}$-semilinear space over $\mathcal{L}$, where

$$V^n(L) = \{(a_1, a_2, \ldots, a_n) : a_i \in L, i \in \underline{n}\}$$

and $\mathbf{0}_{1\times n} = (0, 0, \ldots, 0)$.

(b) Let $X \neq \emptyset$, $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be a semiring. Put $A = L^X = \{f : f : X \to L\}$ and for all $f, g \in A$ define

$$\mathbf{f}(x) +_A \mathbf{f}(y) = \mathbf{f}(x) + \mathbf{f}(y),$$

$$r{*}\mathbf{f}(x) = r \cdot \mathbf{f}(x), \quad \text{for all } x \in X, r \in L.$$

Let $\mathbf{0}_A$ be the function $\mathbf{0}_A : x \mapsto 0$. Then $\mathcal{A} = (L, +, \cdot, 0, 1; *; A, +_A, \mathbf{0}_A)$ is an $\mathcal{L}$-semilinear space.

From now on, without causing confusion we use $r\mathbf{a}$ instead of $r{*}\mathbf{a}$ for all $r \in L$ and $\mathbf{a} \in A$ in an $\mathcal{L}$-semilinear space $\langle L, +, \cdot, 0, 1; *; A, +_A, 0_A\rangle$.

Definition 2.4 (Di Nola et al. [7]). Let $\langle L, +, \cdot, 0, 1; *; A, +_A, 0_A\rangle$ be an $\mathcal{L}$-semilinear space. The expression

$$\lambda_1\mathbf{a}_1 +_A \cdots +_A \lambda_n\mathbf{a}_n,$$

where $\lambda_1, \ldots, \lambda_n \in L$ are scalars is called a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in A$.

Definition 2.5 (Di Nola et al. [7]). In $\mathcal{L}$-semilinear space, a single vector $\mathbf{a}$ is linearly independent. Vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $n \geqslant 2$, are linearly independent if none of them can be represented by a linear combination of the others. Otherwise, we say that vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly dependent. An infinite set of vectors is linearly independent if any finite subset of it is linearly independent.

Notice that alternative concepts related to linear dependence and linear independence in semilinear spaces or semimodules have been studied by other authors (see e.g. [4–6,8,9]).

A nonempty subset G of an $\mathcal{L}$-semilinear space is called a set of generators if every element of the $\mathcal{L}$-semilinear space is a linear combination of elements in G (see e.g. [6]). Let S be a set of generators of $\mathcal{L}$-semilinear space $\mathcal{A}$. Then denote as $\mathcal{A} = \langle S\rangle$.

Definition 2.6 (Golan [8]). A linearly independent set of generators of an $\mathcal{L}$-semilinear space $\mathcal{A}$ is called a basis of $\mathcal{A}$.

Remark 2.1. In general, the cardinality of a basis is not unique (see e.g. [7,15,17,18]).

We denote by $M_{m\times n}(L)$ the set of all $m \times n$ matrices over a semiring $\mathcal{L}=\langle L, +, \cdot, 0, 1\rangle$. Especially let $M_n(L) = M_{n\times n}(L)$. Given $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n} \in M_{m\times n}(L)$ and $C = (c_{ij})_{n\times l} \in M_{n\times l}(L)$, we define that

$$A + B = (a_{ij} + b_{ij})_{m\times n},$$

$$AC = \left(\sum_{k\in \underline{n}} a_{ik} \cdot c_{kj}\right)_{m\times l},$$

$$\lambda A = (\lambda a_{ij})_{m\times n} \quad \text{for all } \lambda \in L.$$

Then $\langle M_n(L), +, \cdot, O_n, I_n\rangle$ is a semiring with

$$O_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Some notions can also be extended to elements of $M_{m\times n}(L)$ in the same way as in classical linear algebra, such as a permutation matrix, a diagonal matrix, etc., we omit them.

Definition 2.7 (Golan [8]). An element $a \in L$ is called invertible in semiring $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ if there exists an element $b \in L$ such that $ab = ba = 1$. Such element b is said to be an inverse of a, and denoted a^{-1}. Let $U(L)$ denote the set of all invertible elements in the semiring $\mathcal{L}$.

Definition 2.8 (Golan [8]). A matrix A in $M_n(L)$ is said to be right invertible (resp. left invertible) if $AB = I_n$ (resp. $BA = I_n$) for some $B \in M_n(L)$. The matrix B is called a right inverse (resp. left inverse) of A in $M_n(L)$. If A is both right and left invertible in $M_n(L)$, then it is called invertible.

Let $A \in M_n(L)$. Denote by P the set of all the permutations of the set $\{1, 2, \ldots, n\}$. The determinant of A, in symbols $det(A)$, is defined by:

$$det(A) = \sum_{\sigma \in P} a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

From the above definition, we know that $det(A) = det(A^T)$ and the following lemma is true.

Lemma 2.1 (Poplin et al. [13]). *Let $A \in M_n(L)$. If there exists an index $k \in \underline{n}$ such that $a_{jk} = 0$ for all $j \in \underline{n}$, then $det(A) = 0$.*

In what follows, we always suppose that $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ is a commutative zerosumfree semiring. The following two lemmas will be used later.

Lemma 2.2 (Tan [16]). *Let $A \in M_n(L)$. Then the following statements are equivalent:*

(1) *A is right invertible;*

(2) *A is left invertible;*

(3) *A is invertible;*

(4) *AA^T is an invertible diagonal matrix;*

(5) *A^TA is an invertible diagonal matrix.*

Lemma 2.3 (Tan [16]). *Let $A, B \in M_n(L)$. If A is invertible, then $det(AB) = det(A) \cdot det(B)$ and $det(BA) = det(B) \cdot det(A)$.*

3. The characterizations of $\mathcal{L}$-semilinear spaces which are generated by standard orthogonal vectors

In this section, we shall investigate some characterizations of standard orthogonal vectors in $\mathcal{V}_n$ and the conditions that a set of vectors is a basis of a semilinear subspace which is generated by standard orthogonal vectors.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{V}_n$. The inner product of $\mathbf{x}$ and $\mathbf{y}$, denoted by $(\mathbf{x}, \mathbf{y})$, is the scalar obtained by multiplying corresponding components and adding the resulting products:

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Note that by the definition of inner product of two vectors, it is easy to see that $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$ and $k(\mathbf{x}, \mathbf{y}) = (k\mathbf{x}, \mathbf{y}) = (\mathbf{x}, k\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}_n$ and $k \in L$.

For all $a, b \in L$, if $r = a + b$ implies that $r = a$ or $r = b$, then r is called an additive irreducible element of semiring $\mathcal{L}$ (see [17]). The following definition is taken from [15].

Definition 3.1. Two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathcal{V}_n$ are said to be orthogonal if $(\mathbf{x}, \mathbf{y}) = 0$. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathcal{V}_n$. If $(\mathbf{x}_i, \mathbf{x}_j) = 0$ and $(\mathbf{x}_i, \mathbf{x}_i) \in U(L)$ for all $i \neq j$, $i, j \in \underline{m}$, then the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is said to be standard orthogonal.

We shall give some examples of sets of standard orthogonal vectors as below.

Example 3.1. In $\mathcal{V}_n$, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is standard orthogonal, where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Example 3.2. Let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be the second semiring in Example 2.3. Then $U(L) = L \setminus \{0\}$ and in $\mathcal{V}_3$, it is obvious that $\left\{\mathbf{a}_1 = \begin{pmatrix} a_1 \\ 0 \\ a_2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ a_3 \\ 0 \end{pmatrix}\right\}$ is standard orthogonal, where $a_i \in L$, $i \in \underline{3}$ and $a_3 \neq 0$, a_1 or $a_2 \neq 0$.

Example 3.3. Let $\mathcal{L} = \langle [0, 1], +, \cdot, 0, 1\rangle$ be the first semiring in Example 2.3. Then it is easy to see that $U(L) = \{1\}$ and 1 is an additive irreducible element. In $\mathcal{V}_3$, it is obvious that $\left\{\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ with $a \in L$ is standard orthogonal.

The following characterization is similar to that of linear algebra.

Theorem 3.1. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s, \mathbf{a}$ are linearly dependent, then $\mathbf{a}$ can be uniquely represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$.*

Proof. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s, \mathbf{a}$ are linearly dependent, then from Definition 2.5, either $\mathbf{x}_i$, $i \in \underline{s}$, or $\mathbf{a}$ can be represented by a linear combination of the other vectors. Suppose that there exist $k, l_j \in L$ such that

$$\mathbf{x}_i = k\mathbf{a} + \sum_{j=1, j\neq i}^{s} l_j \mathbf{x}_j. \tag{1}$$

Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal, with Definition 3.1 equality (1) implies that $(\mathbf{x}_i, \mathbf{x}_i) = k(\mathbf{x}_i, \mathbf{a}) \in U(L)$. Thus $1 = k(\mathbf{x}_i, \mathbf{a})(\mathbf{x}_i, \mathbf{x}_i)^{-1}$, i.e., $k \in U(L)$ by Definition 2.7. On the other hand, from equality (1) we have

$$0 = (\mathbf{x}_t, \mathbf{x}_i) = k(\mathbf{x}_t, \mathbf{a}) + l_t(\mathbf{x}_t, \mathbf{x}_t), \quad t \in \underline{s},\ t \neq i,$$

i.e.,

$$0 = k(\mathbf{x}_t, \mathbf{a}) + l_t(\mathbf{x}_t, \mathbf{x}_t), \quad t \in \underline{s},\ t \neq i.$$

Hence $l_t(\mathbf{x}_t, \mathbf{x}_t) = 0$ since $\mathcal{L}$ is a commutative zerosumfree semiring, which implies that $l_t = 0$, $t \neq i$, $t \in \underline{s}$ since $(\mathbf{x}_t, \mathbf{x}_t) \in U(L)$. Consequently, equality (1) means that $\mathbf{a} = k^{-1}\mathbf{x}_i$, i.e., $\mathbf{a}$ can be represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$.

Now, let

$$\mathbf{a} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_s\mathbf{x}_s = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_s\mathbf{x}_s$$

for $a_i, b_i \in L$ with $i \in \underline{s}$. Then $a_i(\mathbf{x}_i, \mathbf{x}_i) = b_i(\mathbf{x}_i, \mathbf{x}_i)$ with $i \in \underline{s}$, i.e., $a_i = b_i$ by Definition 3.1. This concludes the proof. □

Remark 3.1. In general, in Theorem 3.1, the condition of standard orthogonality cannot be omitted. For example, in $\mathcal{V}_3$ over semiring $\mathcal{L} = \langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ with the usual operations of addition $+$ and multiplication $\cdot$ of integers. It is clear that $\mathbf{a}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent, and $\mathbf{a}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are linearly dependent. However, $\mathbf{b}$ cannot be represented by a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Next, we shall define the equivalent sets of vectors in $\mathcal{L}$-semilinear space as follows.

Definition 3.2. In $\mathcal{L}$-semilinear space, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ be two sets of vectors. If every $\mathbf{x}_i$, $i \in \underline{s}$, can be represented by a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$ and every $\mathbf{y}_j$, $j \in \underline{p}$, can be represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$, then the two sets of vectors are said to be equivalent.

Remark 3.2. In general, the number of vectors in two equivalent sets of vectors may be different. For example, let $\mathcal{L}$=$\langle L, +, \cdot, 0, 1 \rangle$ be the last semiring in Example 2.3. Then in $\mathcal{L}$-semilinear space $\mathcal{V}_2$, it is obvious that $\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \end{pmatrix} \right\}$ are equivalent with different number of vectors.

However, we have the following theorem.

Theorem 3.2. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ be two sets of standard orthogonal vectors in $\mathcal{V}_n$. If they are equivalent, then $s = p$.*

Proof. If $s \neq p$, then either $s > p$ or $s < p$. Suppose that $s > p$. Let $\mathbf{y}_i = \sum_{j=1}^{s} a_{ij}\mathbf{x}_j$ with $a_{ij} \in L$ for any $i \in \underline{p}$ by Definition 3.2. Thus

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)A$$

with

$$A = \begin{pmatrix} a_{11} & \cdots & a_{p1} \\ \cdots & \cdots & \cdots \\ a_{1s} & \cdots & a_{ps} \end{pmatrix}.$$

In a similar way, we can let

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)B$$

with

$$B = \begin{pmatrix} b_{11} & \cdots & b_{s1} \\ \cdots & \cdots & \cdots \\ b_{1p} & \cdots & b_{sp} \end{pmatrix}, \quad b_{ji} \in L, \ i \in \underline{p}, \ j \in \underline{s}.$$

Therefore,

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)AB,$$

i.e., $\mathbf{x}_i = \sum\limits_{k=1}^{s} (\sum\limits_{l=1}^{p} a_{lk} b_{il})\mathbf{x}_k, i \in \underline{s}$, it follows from Theorem 3.1 that $AB = I_s$ since $\mathbf{x}_i = 0\mathbf{x}_1 + \cdots + 0\mathbf{x}_{i-1} + 1\mathbf{x}_i + 0\mathbf{x}_{i+1} + \cdots + 0\mathbf{x}_s$ for every $i \in \underline{s}$. Add $s - p$ columns $\mathbf{0}$ to A, and add $s - p$ rows $\mathbf{0}$ to B, then

$$(A\mathbf{0}) \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = I_s.$$

By Definition 2.8 and Lemma 2.2, both $(A\mathbf{0})$ and $\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}$ are invertible matrices. On the other hand, with Lemmas 2.3 and 2.1 we know that

$$det\left((A\mathbf{0}) \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}\right) = det(A\mathbf{0}) \cdot det \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = 0 = det(I_s) = 1,$$

a contradiction. Analogously to above, we can prove that $s < p$ will deduce a contradiction. Therefore, $s = p$. $\square$

Example 3.4. In Theorem 3.2, the condition of standard orthogonality cannot be dropped generally. Let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be the last semiring in Example 2.3. Then in $\mathcal{L}$-semilinear space $\mathcal{V}_2$, it is clear that $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\}$ are equivalent with different number of vectors, and $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is a set of standard orthogonal, but $\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\}$ is not.

In what follows, we shall study the $\mathcal{L}$-semilinear subspace of $\mathcal{V}_n$ which is defined as below.

Definition 3.3. An $\mathcal{L}$-semilinear subspace of $\mathcal{L}$-semilinear space $\mathcal{A}$ is a subset $\mathcal{W}$ of $\mathcal{A}$ such that for $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{W}$ we have $a\mathbf{v}_1 \in \mathcal{W}$ and $\mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{W}$ for all $a \in L$.

Notice that an $\mathcal{L}$-semilinear space $\mathcal{V}_n$ and its subspace are quite different. For instance, over semiring as Fig. 1 (just replace the addition and the multiplication in Definition 2.1 by the join and meet in the lattice as Fig. 1, respectively), from Theorem 3.1 of [15] we know that each basis of $\mathcal{V}_2$ has the

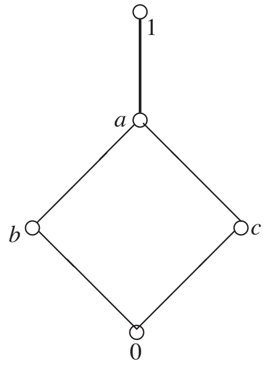

Fig. 1.

same number of elements since 1 is an additive irreducible element. However, by Definitions 2.5 and 3.2 we know that $\left\{\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}\right\}$ are equivalent and linearly independent. Then from Definition 2.6 both $\left\{\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}\right\}$ are bases of the $\mathcal{L}$-semilinear subspace $\left\langle \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right\rangle$ of $\mathcal{V}_2$ with different number of vectors.

Following the proof of Theorem 3.2, the next proposition is true.

Proposition 3.1. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$ and $\mathcal{W} = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s \rangle$. Then the number of elements in every basis of $\mathcal{W}$ is no less than s.*

Note that by Definition 3.3, $\mathcal{W}$ in Proposition 3.1 is an $\mathcal{L}$-semilinear subspace of $\mathcal{V}_n$. In order to investigate the $\mathcal{L}$-semilinear subspace of $\mathcal{V}_n$, the following two lemmas are needed.

Lemma 3.1 (Shu and Wang)**.** *Let $A \in M_n(L)$ and 1 be an additive irreducible element. Then A is invertible if and only if there exists a permutation matrix $P \in M_n(L)$ such that PA is an invertible diagonal matrix.*

Lemma 3.2 (Shu and Wang [15])**.** *A standard orthogonal set is linearly independent.*

Theorem 3.3. *Let $\mathcal{W} = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s \rangle$ with $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ a set of standard orthogonal vectors in $\mathcal{V}_n$. If 1 is an additive irreducible element, and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p \in \mathcal{W}$, then the follow statements are equivalent:*

(1) *$\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is basis of $\mathcal{W}$;*

(2) *$\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is standard orthogonal and $p = s$.*

Proof. (1) $\Rightarrow$ (2). From Definitions 2.6 and 3.2, we know that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ are equivalent, therefore, there exists $A = (a_{ij}) \in M_{p\times s}(L)$ such that

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)A. \tag{2}$$

Using Lemma 3.2 and Definition 2.6 we know that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is a basis of $\mathcal{W}$. Thus for all $\mathbf{y}, \mathbf{z} \in \mathcal{W}$ we can let $\mathbf{y} = \sum_{j=1}^{s} l_j \mathbf{x}_j$ and $\mathbf{z} = \sum_{j=1}^{s} k_j \mathbf{x}_j$ with $l_j, k_j \in L$. For every $i \in \underline{s}$, if $\mathbf{x}_i = \mathbf{y} + \mathbf{z}$ then $\mathbf{x}_i = \sum_{j=1}^{s} (l_j + k_j)\mathbf{x}_j$. From Theorem 3.1 we have that $l_j + k_j = 0$ for all $j \in \underline{s} \setminus \{i\}$, i.e., $l_j = k_j = 0$

since $\mathcal{L}$ is a zerosumfree semiring, and $1 = l_i + k_i$ which means that $l_i = 1$ or $k_i = 1$ since 1 is an additive irreducible element. Therefore, $\mathbf{x}_i = \mathbf{y}$ or $\mathbf{x}_i = \mathbf{z}$. Arguing as above, from Eq. (2), we have that for every $\mathbf{x}_i$, $i \in \underline{s}$, there exists an index $j \in \underline{p}$ such that $\mathbf{x}_i = a_{ji}\mathbf{y}_j$. Then by Definition 3.1 $(\mathbf{x}_i, \mathbf{x}_i) = a_{ji}a_{ji}(\mathbf{y}_j, \mathbf{y}_j) \in U(L)$, i.e., $a_{ji} \in U(L)$. Therefore, $\mathbf{y}_j = a_{ji}^{-1}\mathbf{x}_i$ and $p = s$ which imply that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ is also standard orthogonal.

(2) $\Rightarrow$ (1). Let $\mathbf{y}_i = \sum_{j=1}^{s} b_{ij}\mathbf{x}_j$ with $b_{ij} \in L$ for any $i \in \underline{s}$ by Definition 2.4 and the definition of generators. Thus

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)B \tag{3}$$

with

$$B = \begin{pmatrix} b_{11} & \cdots & b_{s1} \\ \cdots & \cdots & \cdots \\ b_{1s} & \cdots & b_{ss} \end{pmatrix}.$$

Since $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ is standard orthogonal, we have

$$0 = (\mathbf{y}_i, \mathbf{y}_j) = \left(\sum_{k=1}^{s} b_{ik}\mathbf{x}_k, \sum_{t=1}^{s} b_{jt}\mathbf{x}_t\right) = \sum_{t=1}^{s}\sum_{k=1}^{s} b_{ik}b_{jt}(\mathbf{x}_k, \mathbf{x}_t), \quad i, j \in \underline{s},\ i \neq j, \tag{4}$$

and

$$(\mathbf{y}_i, \mathbf{y}_i) = \left(\sum_{k=1}^{s} b_{ik}\mathbf{x}_k, \sum_{t=1}^{s} b_{it}\mathbf{x}_t\right) = \sum_{t=1}^{s}\sum_{k=1}^{s} b_{ik}b_{it}(\mathbf{x}_k, \mathbf{x}_t) \in U(L), \quad i \in \underline{s}. \tag{5}$$

Then by formula (5) we know that $\sum_{k=1}^{s} b_{ik}b_{ik}(\mathbf{x}_k, \mathbf{x}_k) \in U(L), i \in \underline{s}$. Let $\sum_{k=1}^{s} b_{ik}b_{ik}(\mathbf{x}_k, \mathbf{x}_k) = d \in U(L)$. Then $d^{-1}\sum_{k=1}^{s} b_{ik}b_{ik}(\mathbf{x}_k, \mathbf{x}_k) = 1$, thus for every $i \in \underline{s}$ there exists an index $l \in \underline{s}$ such that $d^{-1}b_{il}b_{il}(\mathbf{x}_l, \mathbf{x}_l) = 1$ since 1 is an additive irreducible element, therefore, by Definition 2.7 we have $b_{il} \in U(L)$, which means that every column of B has an invertible element. On the other hand, by formula (4) we have $b_{il}b_{jt}(\mathbf{x}_l, \mathbf{x}_t) = 0$ with $i, j, t \in \underline{s}$, $i \neq j$ since semiring $\mathcal{L}$ is zerosumfree. In particular, $b_{il}b_{jl}(\mathbf{x}_l, \mathbf{x}_l) = 0$ with $i, j \in \underline{s}, i \neq j$. At the same time, $(\mathbf{x}_l, \mathbf{x}_l) \in U(L)$ since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal. Therefore, $b_{il}b_{jl} = 0$ with $i, j \in \underline{s}, i \neq j$, thus $b_{jl} = 0$ which results that every row of B has exact one invertible element and the others are zeroes. Therefore, it follows from Lemma 3.1 that B is invertible. Consequently, $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ are equivalent. Again, from Lemma 3.2 we know that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s$ are linearly independent since they are standard orthogonal. Therefore, $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ is basis of $\mathcal{W}$. □

Definition 3.4 (Shu and Wang [15]). If each basis of $\mathcal{L}$-semilinear space $\mathcal{A}$ has the same number of elements, then we call the number of vectors in each basis a dimension of $\mathcal{A}$, in symbols $dim(\mathcal{A})$.

Below, according to Definition 3.4, if every basis of $\mathcal{L}$-semilinear subspace $\mathcal{W}$ has the same number of elements then we can define its number the dimension of $\mathcal{W}$, in symbols $dim(\mathcal{W})$. Due to Theorem 3.3 and its proof, we have

Corollary 3.1. *Let $\mathcal{W} = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\rangle$ with $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ a set of standard orthogonal vectors in $\mathcal{V}_n$. If $U(L) = \{1\}$ and 1 is an additive irreducible element, then:*

(i) *$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is the unique basis of $\mathcal{W}$;*
(ii) *$dim(\mathcal{W}) = s$.*

Theorem 3.4. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ be equivalent and linearly independent in $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal if and only if so is $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$.*

Proof. Sufficiency. From the proof of Theorem 3.2, we have that $s \leqslant p$. Let $\mathbf{y}_i = \sum_{j=1}^{s} a_{ij}\mathbf{x}_j$ with $a_{ij} \in L$ for any $i \in \underline{p}$ by Definition 3.2. Thus

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)A \tag{6}$$

with

$$A = \begin{pmatrix} a_{11} & \cdots & a_{p1} \\ \cdots & \cdots & \cdots \\ a_{1s} & \cdots & a_{ps} \end{pmatrix}.$$

Since $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$ are linearly independent, then every column of A has nonzero element. In a similar way, we can let

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)B$$

with

$$B = \begin{pmatrix} b_{11} & \cdots & b_{s1} \\ \cdots & \cdots & \cdots \\ b_{1p} & \cdots & b_{sp} \end{pmatrix}, \quad b_{ji} \in L,\ i \in \underline{p},\ j \in \underline{s},$$

and every row of B has nonzero element. Therefore,

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)AB. \tag{7}$$

Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal, then

$$0 = (\mathbf{x}_i, \mathbf{x}_j) = \left(\sum_{k=1}^{p} b_{ik}\mathbf{y}_k, \sum_{t=1}^{p} b_{jt}\mathbf{y}_t\right) = \sum_{t=1}^{p}\sum_{k=1}^{p} b_{ik}b_{jt}(\mathbf{y}_k, \mathbf{y}_t), \quad i, j \in \underline{s},\ i \neq j.$$

Then $b_{ik}b_{jt}(\mathbf{y}_k, \mathbf{y}_t) = 0$ with $i, j \in \underline{s}$, $i \neq j$, $k, t \in \underline{p}$, since semiring $\mathcal{L}$ is zerosumfree. In particular, $b_{ik}b_{jk}(\mathbf{y}_k, \mathbf{y}_k) = 0$ with $i, j \in \underline{s}$, $i \neq j$, $k \in \underline{p}$. On the other hand, we know that $(\mathbf{y}_k, \mathbf{y}_k) \neq 0$ with $k \in \underline{p}$ since $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is linearly independent. Therefore, $b_{ik}b_{jk} = 0$ with $i, j \in \underline{s}$, $i \neq j$, $k \in \underline{p}$ since $U(L) = L \setminus \{0\}$, which means that every row of B has exact one nonzero element, say

$$b_{i_1 1}, b_{i_2 2}, \ldots, b_{i_p p} \neq 0, \quad i_1, i_2, \ldots, i_p \in \underline{s}. \tag{8}$$

Again, with Theorem 3.1 and equality (7) we have $AB = I_s$, and

$$\mathbf{x}_i = \sum_{k=1}^{p} b_{ik}\mathbf{y}_k = \sum_{k=1}^{p} b_{ik}\left(\sum_{t=1}^{s} a_{kt}\mathbf{x}_t\right) = \sum_{t=1}^{s}\left(\sum_{k=1}^{p} b_{ik}a_{kt}\mathbf{x}_t\right).$$

Thus

$$\sum_{k=1}^{p} b_{ik}a_{ki} = 1 \quad \text{and} \quad \sum_{k=1}^{p} b_{ik}a_{kt} = 0, \quad t \neq i. \tag{9}$$

Equality (9) implies that $b_{i_k k}a_{kt} = 0$, $t \neq i_k$, $k \in \underline{p}$, since semiring $\mathcal{L}$ is zerosumfree. Therefore, with (8) we have $a_{kt} = 0$, $t \neq i_k$, $k \in \underline{p}$, i.e., every column of A has exact one nonzero element. If $s \neq p$, then $s < p$ and there exists an index $j \in \underline{s}$ such that $a_{hj}, a_{lj} \neq 0$, $h \neq l$, $h, l \in \underline{p}$. In sequel, equality (6) implies that $\mathbf{y}_h = a_{hj}\mathbf{x}_j$ and $\mathbf{y}_l = a_{lj}\mathbf{x}_j$, that is to say that $\{\mathbf{y}_h, \mathbf{y}_l\}$ is linearly dependent, a contradiction. Therefore, $s = p$. Thus both every row and column of A have exact one nonzero element. Consequently, with (6) we know that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is standard orthogonal since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal. Arguing as above, we can prove the necessity. □

Note that in Theorem 3.4 the condition of $U(L) = L\setminus\{0\}$ cannot be deleted generally. For instance, let $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be the last semiring in Example 2.3. Then it is easy to see that $U(L) \neq L\setminus\{0\}$, and in $\mathcal{L}$-semilinear space $\mathcal{V}_2$, $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}\right\}$ are equivalent and linearly independent. However, $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a set of standard orthogonal, but $\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}\right\}$ is not.

By the proof of Theorem 3.4 and Definition 3.4 we deduce:

Corollary 3.2. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then $dim(\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\rangle) = s$.*

Lemma 3.3. *If $U(L) = L \setminus \{0\}$, then $A \in M_n(L)$ is invertible if and only if there exists a permutation matrix $Q \in M_n(L)$ such that QA is an invertible diagonal matrix.*

Proof. Suppose that A is invertible and $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is the set of column vectors of A. Let B be the inverse of A. Then $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)B = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$. Thus $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are equivalent. From Corollary 3.2 we know that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly independent, further, by Theorem 3.4 we have that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is standard orthogonal since $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is standard orthogonal. Thus by the proof of Theorem 3.4 we know that every row of B has exact one nonzero element. In a similar way, we know that every row of A has exact one nonzero element. Therefore, there exists a permutation matrix $Q \in M_n(L)$ such that QA is an invertible diagonal matrix. The converse part directly comes from the hypothesis. □

Theorem 3.5. *Let $\mathcal{W} = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\rangle$ with $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ a set of standard orthogonal vectors in $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$ and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p \in \mathcal{W}$, then the following statements are equivalent:*

(1) *$\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is basis of $\mathcal{W}$;*
(2) *$\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is standard orthogonal and $p = s$.*

Proof. (1) ⇒ (2). It is due to Lemma 3.2, Definition 2.6 and Theorem 3.4.

(2) ⇒ (1). Let $\mathbf{y}_i = \sum_{j=1}^{s} a_{ij}\mathbf{x}_j$ with $a_{ij} \in L$ for any $i \in \underline{s}$ by Definition 2.4 and the definition of generators. Thus

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)A \tag{10}$$

with

$$A = \begin{pmatrix} a_{11} & \cdots & a_{s1} \\ \cdots & \cdots & \cdots \\ a_{1s} & \cdots & a_{ss} \end{pmatrix}.$$

It is easy to prove that both every row and column of A have exact one nonzero element. Thus from Lemma 3.3 we have that A is invertible since $U(L) = L \setminus \{0\}$, it follows from Eq. (10) that every element of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ can be represented by a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s$. Therefore, $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ is a basis of $\mathcal{W}$ by Lemma 3.2 and Definition 2.6. □

In classical linear algebra, every set of linearly independent vectors can be orthogonalized. However, by Theorem 3.4 we have:

Theorem 3.6. *Let $U(L) = L \setminus \{0\}$. Then a set of linearly independent nonstandard orthogonal vectors in $\mathcal{V}_n$ cannot be orthogonalized if it has at least two nonzero vectors.*

Lemma 3.4 (Shu and Wang [15]). *In $\mathcal{L}$-semilinear space $\mathcal{V}_n$, if $dim(\mathcal{V}_n) = n$, a set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a basis of $\mathcal{V}_n$ if and only if it is standard orthogonal.*

From Lemma 3.4, we have:

Corollary 3.3. *In $\mathcal{L}$-semilinear space $\mathcal{V}_n$, if $dim(\mathcal{V}_n) = n$, the number of vectors in every standard orthogonal set is no more than n.*

Theorem 3.4 and Corollary 3.3 imply the following statement holds:

Corollary 3.4. *Let $\mathcal{W}$ be an $\mathcal{L}$-semilinear subspace of $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$ and there exists a set of standard orthogonal vectors which is a basis of $\mathcal{W}$, then $dim(\mathcal{W}) \leqslant dim(\mathcal{V}_n)$.*

Let $A \in M_{m\times p}(L)$. The row space of A is the $\mathcal{L}$-semilinear subspace of $\mathcal{V}_p$ generated by its rows, and the column space of A is the $\mathcal{L}$-semilinear subspace of $\mathcal{V}_m$ generated by its columns (see [10]). Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathcal{V}_n$. Symbol $A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m)$ stands for an $n \times m$ matrix with vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ as columns.

In what follows, we shall relate a set of standard orthogonal vectors with the dimension of row (resp. column) space and the factor rank of matrix, respectively.

Theorem 3.7. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then both the row and column spaces of $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ have the same dimension.*

Proof. Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ denote the row vectors of matrix $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$. By Corollary 3.3 we know that $s \leqslant n$. It is clear that every column of $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ has nonzero elements and every row of it has at most one nonzero element since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is standard orthogonal. Thus with Definition 3.1 there is a set of row vectors which is standard orthogonal and its number is s, it is also a basis of row space of matrix $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ since every row vector of matrix $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ is s-dimensional. Therefore, it follows from Theorem 3.4 and Definition 3.4 that both the row and column spaces of $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ have the same dimension. $\square$

Note that in Theorem 3.7, the condition of $U(L) = L \setminus \{0\}$ cannot be deleted generally. For example, over semiring $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ which is the last semiring in Example 2.3, it is clear that $U(L) \neq L \setminus \{0\}$. In matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}$, we know that $\left\{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}\right\}$ is standard orthogonal, $\left\{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \\ 0 \end{pmatrix}\right\}$ are equivalent and linearly independent, and from Definition 2.6 we know that both $\left\{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \\ 0 \end{pmatrix}\right\}$ are bases of the column space of the matrix. Then by Definition 3.4 we cannot define the dimension of the column space of the matrix. Therefore, the row and column spaces of $\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}$ never have the same dimension.

Definition 3.5 (Golan [8]). A matrix $A \in M_{m\times n}(L)$ is said to be of the factor rank k, denoted by $f(A) = k$, if there exist two matrices $B \in M_{m\times k}(L)$ and $C \in M_{k\times n}(L)$ such that $A = BC$ and k is the smallest positive integer such that this factorization exists.

Lemma 3.5 (Zhao and Wang [17]). *Let $A \in M_n(L)$ be an invertible matrix. Then $f(A) = n$.*

Let $A \in M_{n\times m}(L)$. Denote by $A[i_1, \ldots, i_s|j_1, \ldots, j_t]$ the $s \times t$ submatrix of A obtained from A whose (p, q)-entry is equal to $a_{i_p j_q}$, where $p \in \underline{s}$, $q \in \underline{t}$, $\{i_1, \ldots, i_s\} \subseteq \underline{n}$ and $\{j_1, \ldots, j_t\} \subseteq \underline{m}$ with $i_p \neq i_q$ and $j_p \neq j_q$ $(p \neq q)$.

Lemma 3.6 (Pshenitsyna [14]). *Let $A \in M_{n\times m}(L)$ and $f(A) = k$. Then for all $\{i_1, \ldots, i_s\} \subseteq \underline{n}$ and $\{j_1, \ldots, j_t\} \subseteq \underline{m}$ with $i_p \neq i_q$ $(p \neq q)$ and $j_p \neq j_q$ $(p \neq q)$, the inequality $f(A[i_1, \ldots, i_s \mid j_1, \ldots, j_t]) \leqslant f(A)$ is true.*

Theorem 3.8. *Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then $f(A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)) = s$.*

Proof. Let

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{n1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{n2} \end{pmatrix}, \ldots, \mathbf{a}_s = \begin{pmatrix} a_{1s} \\ a_{2s} \\ \cdots \\ a_{ns} \end{pmatrix}.$$

Then from Definition 3.1 we have

$$a_{ij}a_{ik} = 0, \quad i \in \underline{n}, \quad k \neq j, k, j \in \underline{s} \tag{11}$$

and

$$\sum_{i=1}^{n} a_{ij}^2 \in U(L), \quad j \in \underline{s}. \tag{12}$$

Formulas (12) and (11) mean that $A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$ has a submatrix $A[i_1, \ldots, i_s \mid 1, \ldots, s]$ with $a_{i_j,j} \neq 0$ and $a_{i_j,k} = 0$, $k \neq j$, $k, j \in \underline{s}$, $i_j \in \underline{n}$. With Lemma 3.3 it is easy to see that $A[i_1, \ldots, i_s \mid 1, \ldots, s]$ is invertible since $a_{i_j j} \in U(L)$ with $j \in \underline{s}$. Therefore from Lemmas 3.5 and 3.6,

$$s = f(A[i_1, \ldots, i_s \mid 1, \ldots, s]) \leqslant f(A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)) \leqslant s,$$

i.e., $f(A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)) = s$. □

Both the proofs of Theorems 3.7 and 3.8 imply the following statement holds:

Corollary 3.5. *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then every row of $A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)$ has at most one nonzero element.*

4. Generalized Kronecker–Capelli theorem

In this section, we shall introduce a generalized notion of a rank of matrix, and prove that the Kronecker–Capelli theorem for a matrix equation is valid over a commutative zerosumfree semiring.

Definition 4.1. Let $A \in M_{n\times m}(L)$. We call the dimension (if it exists) of column (resp. row) space of A the column (resp. row) rank of A, denoted by $r_c(A)$ (resp. $r_r(A)$). If $r_c(A) = r_r(A)$, then we say that the rank of A, written $r(A)$, is $r_c(A)$ or $r_r(A)$.

Remark 4.1. In Definition 4.1, the definition of rank of matrix is quite different from that of [12] (see Definition 2.3 of [12]).

Example 4.1. Let semiring $\mathcal{L} = \langle L, +, \cdot, 0, 1\rangle$ be the last semiring in Example 2.3. It is clear that the column rank of matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}$ does not exist (see the example between Theorem 3.7 and Definition 3.5 for detail), and so does not the rank of the matrix by Definition 4.1. However, the rank of the matrix is 2 in the sense of the definition of rank of matrix in [12].

In order to give another example to explain Remark 4.1, we need the following lemma:

Lemma 4.1. *Let semiring $\mathcal{L} = \langle \mathbb{N}, +, \cdot, 0, 1\rangle$ be the semiring in Example 2.2, and $\mathcal{W}$ be an $\mathcal{L}$-semilinear subspace of $\mathcal{V}_n$. Then each basis of $\mathcal{W}$ has the same number of elements.*

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be a basis of $\mathcal{W}$. First, we shall prove that every $\mathbf{x}_i$, $i \in \underline{s}$, can be uniquely represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$. Indeed, let

$$\mathbf{x}_i = k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_s\mathbf{x}_s \text{ with } k_i \in \mathbb{N}.$$

We know that for every $k_i \in \mathbb{N}$ there exists an element $r \in \mathbb{N}$ such that either $1 + r = k_i$ or $k_i + r = 1$. In the case $1 + r = k_i$, we have

$$\mathbf{x}_i = k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + (1 + r)\mathbf{x}_i + \cdots + k_s\mathbf{x}_s,$$

which follows that

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + r\mathbf{x}_i + \cdots + k_s\mathbf{x}_s = \mathbf{0},$$

then $k_j = r = 0$ for every $j \neq i, j \in \underline{s}$, and $k_i = 1$ since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is a basis of $\mathcal{W}$. In the case $k_i + r = 1$, we surely have that $k_i = 1$ and $k_j = r = 0$ for every $j \neq i, j \in \underline{s}$ since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is a basis of $\mathcal{W}$. Therefore, every $\mathbf{x}_i$, $i \in \underline{s}$, can be uniquely represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$. Now, suppose that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is another basis of $\mathcal{W}$, we shall prove $s = p$. Let $\mathbf{y}_i = \sum_{j=1}^{s} a_{ij}\mathbf{x}_j$ with $a_{ij} \in L$ for every $i \in \underline{p}$. Thus

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)A$$

with

$$A = \begin{pmatrix} a_{11} & \cdots & a_{p1} \\ \cdots & \cdots & \cdots \\ a_{1s} & \cdots & a_{ps} \end{pmatrix}.$$

In a similar way, we can let

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)B$$

with

$$B = \begin{pmatrix} b_{11} & \cdots & b_{s1} \\ \cdots & \cdots & \cdots \\ b_{1p} & \cdots & b_{sp} \end{pmatrix}, \quad b_{ji} \in L, \ i \in \underline{p}, \ j \in \underline{s}.$$

Therefore,

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)AB = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s)I_s,$$

i.e., $AB = I_s$ since every $\mathbf{x}_i, i \in \underline{s}$, can be uniquely represented by a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$. Similar to the corresponding proof of Theorem 3.2, we can prove that $s = p$. $\square$

Example 4.2. Let semiring $\mathcal{L} = \langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ be the semiring in Example 2.2 and

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \in M_4(\mathbb{N}).$$

Then using Lemma 4.1, Definitions 3.4 and 4.1, we have that $r_c(A) = r_r(A) = r(A) = 4$ since both column and row vectors of A are linearly independent, and $r_r(B) = 3 \neq r_c(B) = 4$ since the column vectors of B are linearly independent and the row ones are linearly dependent. However, both ranks of A and B are 3, respectively, in the sense of the definition of rank of matrix in [12].

By Theorem 3.7 and Definition 4.1, we have

Theorem 4.1. *Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ be standard orthogonal in $\mathcal{L}$-semilinear space $\mathcal{V}_n$. If $U(L) = L \setminus \{0\}$, then $r(A(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)) = s$.*

Let $A = (a_{ij}) \in M_{n\times m}(L)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T \in M_{n\times 1}(L)$. Consider the following system of equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1m}x_m = b_1, \\ a_{21}x_1 + \cdots + a_{2m}x_m = b_2, \\ \cdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m = b_n \end{cases} \tag{13}$$

with respect to an unknown vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T \in \mathcal{V}_m$. Denote the column vectors of A by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. Obviously, they are elements of $\mathcal{V}_n$.

Proposition 4.1 (Perfilieva and Kupka [12]). *System (13) is solvable if and only if $\mathbf{b} \in \langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \rangle$.*

The following theorem is the Kronecker–Capelli theorem for system (13) over a commutative zero-sumfree semiring.

Theorem 4.2. *If $U(L) = L \setminus \{0\}$ and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are standard orthogonal, then system (13) is solvable if and only if $r(A) = r(A\mathbf{b})$, where the matrix $A\mathbf{b}$ is equal to A extended by $\mathbf{b}$ as the last column. Moreover, if system (13) is solvable, then it has a unique solution.*

Proof. If $r(A) = r(A\mathbf{b})$, then by Proposition 4.1 and Definition 4.1 we know that system (13) is solvable. So we just need to prove the necessity. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n \in \mathcal{V}^m$ be row vectors of A. If system (13) is solvable, then by Definition 4.1, Corollary 3.2, Theorem 3.7 and its proof, we have $r_c(A) = r_c(A\mathbf{b}) = r_r(A) = m$ and there exists a sequence of vectors $\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \ldots, \mathbf{c}_{i_m} \in \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ which are standard orthogonal. Then by Corollary 3.5 we would assume that

$$\mathbf{c}_{i_1}^T = \begin{pmatrix} a_{i_1 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{c}_{i_2}^T = \begin{pmatrix} 0 \\ a_{i_2 2} \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{c}_{i_m}^T = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{i_m m} \end{pmatrix}$$

with $a_{i_t t} \neq 0$. Let $\mathbf{c}_1^{'}, \mathbf{c}_2^{'}, \ldots, \mathbf{c}_n^{'}$ be the row vectors of $A\mathbf{b}$. Then

$$(\mathbf{c}_{i_1}^{'})^T = \begin{pmatrix} a_{i_1 1} \\ 0 \\ \vdots \\ 0 \\ b_{i_1} \end{pmatrix}, \quad (\mathbf{c}_{i_2}^{'})^T = \begin{pmatrix} 0 \\ a_{i_2 2} \\ \vdots \\ 0 \\ b_{i_2} \end{pmatrix}, \ldots, (\mathbf{c}_{i_m}^{'})^T = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{i_m m} \\ b_{i_m} \end{pmatrix},$$

in which $b_{i_t} \in \{b_1, b_2, \ldots, b_n\}$ with $t \in \underline{m}$. It is obvious that $\mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'}$ are linearly independent. Since system (13) is solvable, and from Theorem 3.1 we know that $(a_{i_1 1}^{-1} b_{i_1}, a_{i_2 2}^{-1} b_{i_2}, \ldots, a_{i_m m}^{-1} b_{i_m})^T$ is its unique solution, it is easy to check that

$$\mathbf{c}_j^{'} = a_{i_1 1}^{-1} a_{j1} \mathbf{c}_{i_1}^{'} + a_{i_2 2}^{-1} a_{j2} \mathbf{c}_{i_2}^{'} + \cdots + a_{i_m m}^{-1} a_{jm} \mathbf{c}_{i_m}^{'}$$

for all $\mathbf{c}_j^{'} \in \{\mathbf{c}_j^{'} : j \in \underline{n}, \mathbf{c}_j^{'} \neq \mathbf{c}_{i_t}^{'}, t \in \underline{m}\}$, i.e., $\langle \mathbf{c}_{i_1}^{'}, \mathbf{c}_{i_2}^{'}, \ldots, \mathbf{c}_{i_m}^{'} \rangle = \langle \mathbf{c}_1^{'}, \mathbf{c}_2^{'}, \ldots, \mathbf{c}_n^{'} \rangle$. Therefore, to complete the proof, it is enough to prove that each basis of $\langle \mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'} \rangle$ has the same number of elements. First, by Definition 2.6 and the proof as above, we know that $\mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'}$ is a basis of $\langle \mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'} \rangle$. Then we shall prove the following statement:

(A) Every $\mathbf{c}_{i_t}^{'}$, $t \in \underline{m}$, can be uniquely represented by a linear combination of $\mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'}$.
In fact, let

$$\mathbf{c}_{i_t}^{'} = k_1 \mathbf{c}_{i_1}^{'} + k_2 \mathbf{c}_{i_2}^{'} + \cdots + k_m \mathbf{c}_{i_m}^{'}, \ k_t \in L, \ t \in \underline{m}.$$

Then

$$\begin{pmatrix} 0 \\ \vdots \\ a_{i_t t} \\ \vdots \\ b_{i_t} \end{pmatrix}^T = k_1 \begin{pmatrix} a_{i_1 1} \\ 0 \\ \vdots \\ 0 \\ b_{i_1} \end{pmatrix}^T + \cdots + k_t \begin{pmatrix} 0 \\ \vdots \\ a_{i_t t} \\ \vdots \\ b_{i_t} \end{pmatrix}^T + \cdots + k_m \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{i_m m} \\ b_{i_m} \end{pmatrix}^T.$$

Thus $k_j = 0$ for all $j \in \underline{m} \setminus \{t\}$, and $k_t = 1$, i.e., every $\mathbf{c}_{i_t}^{'}$, $t \in \underline{m}$, can be uniquely represented by a linear combination of $\mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'}$.

Continuing with the proof of Theorem 4.2, suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ is another basis of $\langle \mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'} \rangle$. Next, we will prove $s = m$. Let $\mathbf{x}_i = \sum_{t=1}^{m} l_{it}\mathbf{c}_{i_t}^{'}$ with $l_{it} \in L$ for any $i \in \underline{s}$. Thus

$$(\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_s^T) = ((\mathbf{c}_{i_1}^{'})^T, (\mathbf{c}_{i_2}^{'})^T, \ldots, (\mathbf{c}_{i_m}^{'})^T)C \tag{14}$$

with

$$C = \begin{pmatrix} l_{11} & \cdots & l_{s1} \\ \cdots & \cdots & \cdots \\ l_{1m} & \cdots & l_{sm} \end{pmatrix}.$$

In a similar way, we can let

$$\left((\mathbf{c}_{i_1}^{'})^T, (\mathbf{c}_{i_2}^{'})^T, \ldots, (\mathbf{c}_{i_m}^{'})^T\right) = \left(\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_s^T\right) D \tag{15}$$

with

$$D = \begin{pmatrix} k_{11} & \cdots & k_{m1} \\ \cdots & \cdots & \cdots \\ k_{1s} & \cdots & k_{ms} \end{pmatrix}, \quad k_{ji} \in L,\ i \in \underline{s},\ j \in \underline{m}.$$

Therefore,

$$\begin{aligned} \left((\mathbf{c}_{i_1}^{'})^T, (\mathbf{c}_{i_2}^{'})^T, \ldots, (\mathbf{c}_{i_m}^{'})^T\right) &= \left((\mathbf{c}_{i_1}^{'})^T, (\mathbf{c}_{i_2}^{'})^T, \ldots, (\mathbf{c}_{i_m}^{'})^T\right) CD \\ &= \left((\mathbf{c}_{i_1}^{'})^T, (\mathbf{c}_{i_2}^{'})^T, \ldots, (\mathbf{c}_{i_m}^{'})^T\right) I_m, \end{aligned}$$

which together with Statement (A) implies that $CD = I_m$. If $m > s$, then add $m - s$ columns $\mathbf{0}$ to C, and $m - s$ rows $\mathbf{0}$ to D. It is similar to the proof of Theorem 3.2 that $m > s$ will deduce a contradiction. Therefore, $m \leqslant s$. On the other hand, suppose that $\mathbf{x}_j^T = \begin{pmatrix} d_{j1} \\ d_{j2} \\ \vdots \\ d_{jm} \\ d_{j,m+1} \end{pmatrix}$ with $j \in \underline{s}$. Then by Eq. (15) we have

$$\begin{cases} k_{t1}d_{11} + k_{t2}d_{21} + \cdots + k_{ts}d_{s1} = 0, \\ \quad \cdots \\ k_{t1}d_{1t} + k_{t2}d_{2t} + \cdots + k_{ts}d_{st} = a_{i_t t}, \\ \quad \cdots \\ k_{t1}d_{1m} + k_{t2}d_{2m} + \cdots + k_{ts}d_{sm} = 0, \\ k_{t1}d_{1,m+1} + k_{t2}d_{2,m+1} + \cdots + k_{ts}d_{s,m+1} = b_{i_t} \end{cases} \tag{16}$$

with $t \in \underline{m}$. Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ is basis of $\langle \mathbf{c}_{i_1}^{'}, \ldots, \mathbf{c}_{i_m}^{'} \rangle$, with Eq. (15) we know that every row of D has at least one nonzero element, i.e., for every $j \in \underline{s}$ there exists an index $h \in \underline{m}$ such that $k_{hj} \neq 0$.

Then by Eq. (16) we know that for every $j \in \underline{s}$, $d_{jq} = 0$ for all $q \in \underline{m} \setminus \{h\}$. Thus for every $j \in \underline{s}$ we would have that

$$\mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ d_{jh} \\ \vdots \\ 0 \\ d_{j,m+1} \end{pmatrix} \quad \text{or} \quad \mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ d_{j,m+1} \end{pmatrix} \quad \text{or} \quad \mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ d_{jh} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad d_{jh},\ d_{j,m+1} \neq 0.$$

If there exists an index $j \in \underline{s}$ such that $\mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ d_{j,m+1} \end{pmatrix}$, then $\mathbf{x}_j$ cannot be represented by a linear combination of $\mathbf{c}'_{i_1}, \mathbf{c}'_{i_2}, \ldots, \mathbf{c}'_{i_m}$, a contradiction. Therefore, for every $j \in \underline{s}$ we exactly have that

$$\mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ d_{jh} \\ \vdots \\ 0 \\ d_{j,m+1} \end{pmatrix} \quad \text{or} \quad \mathbf{x}_j^T = \begin{pmatrix} 0 \\ \vdots \\ d_{jh} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad d_{jh},\ d_{j,m+1} \neq 0.$$

Now suppose that $m < s$. Then we must have that for some $h_0 \in \underline{m}$ there exist two indices $j_1, j_2 \in \underline{s}$ such that

$$\mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ d_{j_1,m+1} \end{pmatrix} \quad \text{or} \quad \mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad d_{j_1 h_0},\ d_{j_1,m+1} \neq 0$$

and

$$\mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ d_{j_2,m+1} \end{pmatrix} \quad \text{or} \quad \mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad d_{j_2 h_0},\ d_{j_2,m+1} \neq 0.$$

We can distinguish four cases:

(i)

$$\mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ d_{j_1,m+1} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ d_{j_2,m+1} \end{pmatrix}.$$

In this case, from Eq. (14) we would have $\mathbf{x}_{j_1}^T = l_{j_1 h_0}(\mathbf{c}_{h_0}^{'})^T$ and $\mathbf{x}_{j_2}^T = l_{j_2 h_0}(\mathbf{c}_{h_0}^{'})^T$ with $l_{j_1 h_0}, l_{j_2 h_0} \neq 0$. Thus $\mathbf{x}_{j_1}^T$ and $\mathbf{x}_{j_2}^T$ are linearly dependent, a contradiction.

(ii)

$$\mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ d_{j_1,m+1} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

In this case, from Eq. (14) we would have $d_{j_1,m+1} = 0$, a contradiction.

(iii)

$$\mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ d_{j_2, m+1} \end{pmatrix}.$$

It suffice to reason as in the case (ii) in order to obtain another contradiction.

(iv)

$$\mathbf{x}_{j_1}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_1 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{j_2}^T = \begin{pmatrix} 0 \\ \vdots \\ d_{j_2 h_0} \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

In this case, it is easy to see that $\mathbf{x}_{j_1}^T$ and $\mathbf{x}_{j_2}^T$ are linearly dependent, a contradiction.

Therefore, $m = s$, which implies that $r(A) = r_r(A\mathbf{b}) = m = r_c(A\mathbf{b})$, it follows from Definition 4.1 that $r(A) = m = r(A\mathbf{b})$. This concludes the proof. □

Remark 4.2. In general, the condition of standard orthogonality in Theorem 4.2 cannot be deleted. For example, in Remark 3.1, let $\mathbf{a}_1, \mathbf{a}_2$ be the column vectors of matrix A. Although $r(A) = r(A\mathbf{b})$, system $A\mathbf{x} = \mathbf{b}$ is not solvable.

Below, let $U(L) = L \setminus \{0\}$ and the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ of A be standard orthogonal. From the proof of Theorem 3.8, if let $A[i_1, i_2, \ldots, i_m \mid 1, 2, \ldots, m]$, $i_j \neq i_k$, $k \neq j$, $k, j \in \underline{m}$ be the invertible submatrix of A. Then by the proof of Theorem 4.2 we have:

Corollary 4.1. *Let $U(L) = L \setminus \{0\}$ and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be standard orthogonal. If system (13) is solvable, then*

$$\mathbf{x} = \left(a_{i_1 1}^{-1} b_{i_1}, a_{i_2 2}^{-1} b_{i_2}, \ldots, a_{i_m m}^{-1} b_{i_m}\right)^T$$

is the unique solution, where $b_{i_t} \in \{b_1, b_2, \ldots, b_n\}$ with $t \in \underline{m}$.

Example 4.3. In system (13), let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \in M_{3\times 2}(\mathbb{R})$ over semiring $\mathcal{L} = \langle \mathbb{R}, +, \cdot, 0, 1 \rangle$ with the usual operations of addition $+$ and multiplication $\cdot$ of integers and $\mathbf{b} = (2, 4, 2)^T$. Then it is obvious that $r(A) = r(A\mathbf{b})$, by Theorem 4.2 and Corollary 4.1 we know that system (13) is solvable and $\mathbf{x} = (2, 2)^T$ is its unique solution.

5. Conclusions

In this contribution, we have given some characterizations of standard orthogonal vectors, have shown some necessary and sufficient conditions that a set of vectors is a basis of an $\mathcal{L}$-semilinear subspace which is generated by standard orthogonal vectors and have proven that the analog of the Kronecker–Capelli theorem is valid in an $\mathcal{L}$-semilinear vector space. It is worth to point out that we just discuss the $\mathcal{L}$-semilinear subspace which is generated by standard orthogonal vectors, and almost all our results rely on the assumption of standard orthogonality. Therefore, it remains open whether there are other conditions under which the similar results hold.

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